Problem 1

Let N be the least positive integer that is both 22 percent less than one integer and 16 percent greater than another integer. Find the remainder when N is divided by 1000.

Problem 2

In a new school, 40 percent of the students are freshmen, 30 percent are sophomores, 20 percent are juniors, and 10 percent are seniors. All freshmen are required to take Latin, and 80 percent of sophomores, 50 percent of the juniors, and 20 percent of the seniors elect to take Latin. The probability that a randomly chosen Latin student is a sophomore is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 3

Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m.

Problem 4

In an isosceles trapezoid, the parallel bases have lengths $\log 3$ and $\log 192$, and the altitude to these bases has length $\log 16$. The perimeter of the trapezoid can be written in the form $\log 2^p 3^q$, where p and q are positive integers. Find $p+q$.

Problem 5

Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected unit squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

Problem 6

Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers a and c. Can you tell me the values of a and c?" After some calculations, Jon says, "There is more than one such polynomial." Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?" Jon says, "There are still two possible values of c." Find the sum of the two possible values of c.

Problem 7

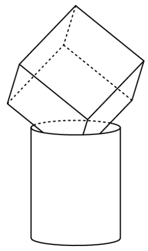
Triangle ABC has side lengths $AB = 12$, $BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial $\text{Area}(PQRS) = \alpha w - \beta \cdot w^2$. Then the coefficient $\beta = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 8

Let a and b be positive integers satisfying $\frac{ab+1}{a+b} < \frac{3}{2}$. The maximum possible value of $\frac{a^3b^3+1}{a^3+b^3}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problem 9

A cylindrical barrel with radius 4 feet and height 10 feet is full of water. A solid cube with side length 8 feet is set into the barrel so that the diagonal of the cube is vertical. The volume of water thus displaced is v cubic feet. Find v^2.



Problem 10

Call a permutation $a_1, a_2, \ldots, a_n$ of the integers $1, 2, \ldots, n$ "quasi-increasing" if $a_k \leq a_{k+1} + 2$ for each $1 \leq k \leq n-1$. For example, 53421 and 14253 are quasi-increasing permutations of the integers $1, 2, 3, 4, 5$, but 45123 is not. Find the number of quasi-increasing permutations of the integers $1, 2, \ldots, 7$.

Problem 11

The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 12

There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

Problem 13

Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = \sum\limits_{k=1}^{n} \sin k$, where k represents radian measure. Find the index of the 100th term for which $a_n < 0$.

Problem 14

Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Problem 15

Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ such that BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ, and the areas of $\triangle DBA$ and $\triangle ACE$ are equal. This common area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.