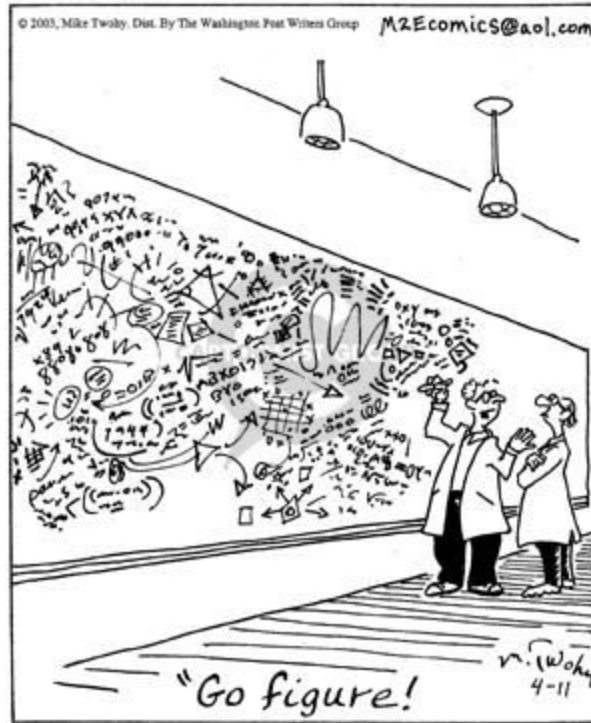


LESSON 1 : VIETA'S FORMULAS INTRODUCTION

- Vieta's Formulas establish relationships between the coefficients in a polynomial and the sum and product of its roots
- Also relates the coefficients to the products of roots taken in groups

1.1: QUADRATIC EQUATIONS**Deriving Vieta's Formula for Quadratics:**

- We begin with a simple quadratic equation,
 $f(x) = ax^2 + bx + c$
- Let the roots of $f(x)$ be denoted by r and s .
- We can then rewrite $f(x)$ in terms of its roots as $f(x) = a(x - r)(x - s)$
 (This is made possible by the Fundamental Theorem of Algebra)
- Expanding this again, we get $f(x) = ax^2 - a(r + s)x + ars$. We know that this must be equivalent to $f(x) = ax^2 + bx + c$.
- The only way for this to hold true is if the coefficients of the corresponding terms are equal.

$ax^2 = ax^2$. This is not really useful for us.

$-a(r + s)x = bx$. This means that $r + s = \frac{-b}{a}$.

$ars = c$. This means that $rs = \frac{c}{a}$.

- We have just derived Vieta's Formulas for quadratic equations. This means that we can now find the sum and product of the roots of quadratic equations very quickly.

Problems:

1. What is the sum of the roots of the quadratic $x^2 - 3x + 5 = 0$?
2. The product of the roots of the equation $kx^2 + 4x + 10$ is 5. What is k ?
3. The sum of the reciprocals of the roots of the quadratic $3x^2 + 4x + 5$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime. Find $p + q$.

$\frac{p}{q}$

TIP: When a question asks you to express a value as $\frac{p}{q}$ where p and q are relatively prime, and then find some value with p and q , such as $p + q$ or pq , it really just wants you to find $\frac{p}{q}$. Once you find this fraction, ensure it is in simplest terms. Then, simply add the numerator and denominator to find $p + q$ or multiply them to find pq .

This question is really just asking us to compute the sum of the reciprocals of the roots of this quadratic. AMC loves to disguise easy questions like this.

4. Express the roots of the equation $x^2 + bx + c$ are r and s . Express $r^2 + s^2$ in terms of b and c .

Solutions:

- Using Vieta's formulas, the sum of the roots of the quadratic is just $\frac{-b}{a}$, or $\frac{-1 \cdot -3}{1} = \boxed{3}$.
- Using Vieta's formulas, the product of the roots of the quadratic is $\frac{c}{a}$, or $\frac{10}{k}$. We know that this is equal to 5, so $\frac{10}{k} = 5$ or $\boxed{k = 2}$.
- Let the two roots be r and s . The problem is asking us to find $\frac{1}{r} + \frac{1}{s}$. We can combine these fractions under a common denominator, and we see that we are looking for $\frac{r+s}{rs}$. We know how to find both the numerator and denominator using Vieta's formulas, so we do that. We find that the numerator is just $\frac{-4}{3}$ and the denominator is $\frac{5}{3}$, so $\frac{p}{q} = \frac{-4}{5}$. This fraction is in simplest form, so we let $p = -4$ and $q = 5$. Thus, $\boxed{p + q = 1}$.
- Recall that after expanding, $(r+s)^2 = r^2 + s^2 + 2rs$. Rewriting this as $r^2 + s^2 = (r+s)^2 - 2rs$, we notice that LHS (left hand side) is our desired quantity, and the RHS (right hand side) can be computed using Vieta's formulas.

$r + s = -b$ by Vieta's, so $(r+s)^2 = (-b)^2 = b^2$. Again, by Vieta's, $rs = c$, so $2rs = 2c$. Combining these two pieces of information, we get that $\boxed{r^2 + s^2 = b^2 - 2c}$.

1.2: CUBIC EQUATIONS

- We begin with a simple cubic equation,

$$f(x) = ax^3 + bx^2 + cx + d.$$

- Let the roots of $f(x)$ be denoted by r, s and t .

- We can then rewrite $f(x)$ in terms of its roots as $f(x) = a(x-r)(x-s)(x-t)$ (This is made possible by the Fundamental Theorem of Algebra)

- The rest of this derivation is left as an exercise for the reader. It is very similar to the derivation of the quadratic Vieta's formulas. Use the space on the next page to attempt this:

- We find that $r + s + t = \frac{-b}{a}$, $rs + st + tr = \frac{c}{a}$, and $rst = \frac{-d}{a}$.

Problems:

1. Let $p(x) = 2x^3 + ax^2 + 38x + c$. If two of the roots are 4 and 6, what is the third root?

2. Let $p(x) = 2x^3 + ax^2 + 38x + c$. If two of the roots are 4 and 6, what is the value of ac ?

3. Let $p(x) = x^3 - 5x^2 + 12x - 19$ have roots r, s, t . What is $\frac{1}{rs} + \frac{1}{st} + \frac{1}{tr}$?

Solutions:

- The only quantity that we know by Vieta's is $rs + tr + ts = \frac{38}{2} = 19$. So we try to use this to find the third root. Let the third root be r . By this equation, $4r + 6r + 24 = 19$, so $10r = -5$ and $\boxed{r = -\frac{1}{2}}$.
- Now that we know all three roots of our polynomial (from the previous question), we use Vieta's to find that $-\frac{a}{2} = r + s + t = \frac{19}{2}$, so $a = -19$. We also know that $-\frac{c}{2} = rst = -12$ so $c = 24$. Thus, $\boxed{ac = -456}$.
- Combining this under a common denominator, we get that $\frac{1}{rs} + \frac{1}{st} + \frac{1}{tr} = \frac{r + s + t}{rst} = \boxed{\frac{5}{19}}$.

1.3: SYMMETRIC SUMS AND GENERAL POLYNOMIAL FORM

- So far we have only computed information about the roots of quadratics and cubics. Vieta's formulas can be extended into all polynomials with a few changes to our derivations. In addition, they can give us even more information about the roots.

1.3.1: SYMMETRIC SUMS

- A **symmetric** polynomial expression is defined as one where swapping any pair of variables leaves the expression unchanged.
- Notice that everything we have computed so far about the roots, i.e. $r + s$, rs , $r + s + t$, rst , and $rs + st + tr$, are all symmetric. Swapping r , s , and t does nothing to them.
- It is very important to realise that an expression like $rs^2 + st^2 + tr^2$ is not symmetric, because switching around the variables r and s results in $sr^2 + rt^2 + ts^2$ which is not the same as the original.
- The expressions we have been computing are called **Elementary Symmetric Sums**.
- The k th Elementary Symmetric Sum of a polynomial is the sum of all possible products of k of the roots. It is notated as S_k .
 - The 1st ESS of a quadratic with roots r and s is $r + s$.
 - We are adding all possible products of 1 root, r and s .
 - The 2nd ESS of a quadratic with roots r and s is rs .

- We are adding all possible products of 2 roots, and there is only one, rs .
 - The 1st ESS of a cubic with roots r , s , and t is $r+s+t$.
 - The 2nd ESS of a cubic with roots r , s , and t is $rs+st+tr$.
 - This is because there are three possible products of 2 roots, rs , st , and tr .
 - The 3rd ESS of a cubic with roots r , s , and t is rst .
- Notice that the 1st ESS of an n th degree polynomial is the sum of the roots, and the n th ESS is the product of the roots.
- It can be proven that using only the Elementary Symmetric Sums, (multiplying, adding, subtracting, raising to integer powers), you can compute any symmetric expression on the roots of a polynomial.

1.3.2: PROVING VIETAS FOR GENERAL POLYNOMIALS

Let

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 && \text{[original form]} \\ &= a_n (x - r_1)(x - r_2) \cdots (x - r_n) && \text{[factored form]} \end{aligned}$$

Expanding this out, we get a polynomial with complicated-looking coefficients.

Our constant term is

$$a_0 = (-1)^n a_n r_1 r_2 \cdots r_n = (-1)^n a_n \prod_{i=1}^n r_i$$

so rearranging our equation, we get

$$\prod_{i=1}^n r_i = (-1)^n \frac{a_0}{a_n} \quad \text{[product of roots or } n\text{th symmetric sum]}$$

This is the product of all the roots.

Similarly, we find

$$\begin{aligned} a_1 &= (-1)^{n-1} a_n [(r_1 r_2 \cdots r_{n-1}) + (r_1 r_2 \cdots r_{n-2} r_n) + \cdots + (r_2 r_3 \cdots r_n)] \\ &= (-1)^{n-1} a_n \sum_{i=1}^n \frac{1}{r_i} \prod_{j=1}^n r_j \end{aligned}$$

so rearranging our equation, we get

$$\sum_{i=1}^n \frac{1}{r_i} \prod_{j=1}^n r_j = (-1)^{n-1} \frac{a_1}{a_n} \quad \text{[(n-1)th symmetric sum]}$$

We can keep continuing. For instance, the sum of all roots is

$$\sum_{i=1}^n r_i = \frac{-a_{n-1}}{a_0} \quad [\text{sum of all roots or 1st symmetric sum}]$$

The sum of the pairwise products is

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^n r_i r_j = \frac{a_{n-2}}{a_n} \quad [2\text{nd symmetric sum}]$$

In general, the k^{th} symmetric sum is (according to Vieta's)

$$S_k = (-1)^k \frac{a_{n-k}}{a_n}.$$

TIP: Ignore the scary notation with all the \sum and \prod s. Only pay attention to the labels and the actual values of the expression. AMC usually doesn't try to confuse you with notation, but there have been a few problems that boil down to understanding confusing notation.

[If there is time at the end of class, example problems will be done. Use the space below to write the problems and take notes.]

